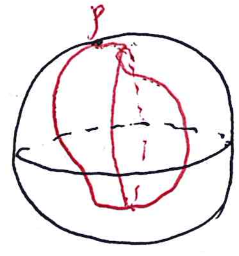
Some Facts for Gaussian Curvature of Surfaces.

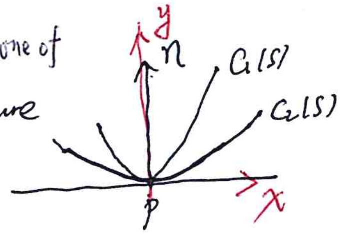
1. **Prop 1** Let S be a (regular) closed orientable surface in $\mathbb{R}^3$, suppose that S is contained in a sphere of radius r, and touching in at a point p on S. Then $K(p) \geq \frac{1}{r^2}$.

Pf: Suppose $\partial_u, \partial_v \in T_pS$ are two principal directions. And the corresponding normal curvature are k_1, k_2, then $K(p) = k_1 k_2$.



We need to show $k_i \geq \frac{1}{r}$.

Choose any vector $X \in T_pS$ s.t. it is one of the eigendirection of n_p. The line of curvature is denoted by $C_1(s) = (x_1(s), y_1(s))$, the corresponding line of curvature on the sphere is $C_2(s)$ (circle) $= (x_2(s), y_2(s))$.



By the assumpsion, we want to show $k_p(C_1(s)) \geq k_p(C_2(s))$.

Since $k_p(C_i(s)) = \ddot{C}_i(s) \cdot n = (\ddot{x}_i(s), \ddot{y}_i(s)) \cdot n = \ddot{y}_i(s)$

and $y_1(s) \approx \frac{1}{2} \ddot{y}_1(0) \cdot s^2$

$y_1(s) \geq y_2(s) \Longrightarrow \ddot{y}_1(0) \geq \ddot{y}_2(0)$

$k_p(C_1(s)) \geq k_p(C_2(s)) = \frac{1}{r}$.

2. **Prop 2** Any closed (regular) orientable surface S in $\mathbb{R}^3$ has a point with positive Gauss curvature. (elliptic point)

Pf: Since S is closed in $\mathbb{R}^3$, it is compact.
$f(x) = |x|^2$ is a continous function in $\mathbb{R}^3$, let p_0 be the maximum pt in S.

$S \subset B(0, |p_0|)$, and $T_{p_0} S = T_{p_0} S^2(0, |p_0|)$.

By Prop 1, $K_{p_0} > 0$.

Method 2: Using Gauss-Bonnet formula. $\iint_S K \, dA = 4\pi$.

3. (Theorem 4.30): Surfaces with the same constant Gaussian curvature are isometric. (Equivalent 1^{st}. f.f.)

<u>Thm 3</u>: Let $f: U \to \mathbb{R}^3$, $\tilde{f}: \tilde{U} \to \mathbb{R}^3$ be surface elements with the same <u>constant</u> Gaussian curvature. Then locally f and $\tilde{f}$ are isometric.

<u>Pf</u>: Let K be the constant Gaussian curvature.

Fix two points $x \in U$, $y \in \tilde{U}$.

Introduce geodesic parallel coordinates (u,v), $u=0$ is also geodesic. (Fermi coordinate).

The 1^{st}- fundamental forms are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & G(u,v) \end{pmatrix}, \quad \tilde{I} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{G}(u,v) \end{pmatrix}$$

with $G(0,v) = 1 = \tilde{G}(0,v)$ (by arc-length parametrization).

By the last tutorial,

$$\frac{\partial^2}{\partial u^2}\sqrt{G} = -K\sqrt{G}, \quad \frac{\partial^2}{\partial u^2}\sqrt{\tilde{G}} = -K\sqrt{\tilde{G}},$$

Given initial conditions $\frac{\partial}{\partial u}\sqrt{G(u,v)}\Big|_{u=0} = 0 = \frac{\partial}{\partial u}\sqrt{\tilde{G}(u,v)}\Big|_{u=0}$ (Fermi coord)

For fixed v, the uniqueness follows because this is an ODE of second order in parameter u.

Then $G(u,v) = \tilde{G}(u,v)$. $I = \tilde{I}$.

Remark: Using Polar geodesic coordinate, one can prove a stronger result: All surfaces of the same constant Gauss curvature are isometric. (Minding's Theorem 1839.)